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Layering Transition in SOS Model with External Magnetic Field

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For the SOS model defined by the Hamiltonian $H(\phi)=\frac{1}{2}\sum_{\langle x,x'\rangle}|\phi_x-\phi_{x'}|+h\sum_x\phi_x$, where ϕ_x, $\phi_{x'}\in\{1,2,...\}$, $h>0$, $x\in\mathbf{Z}^d$, $d\geqslant 2$, it is shown that in the low-temperature region an infinite sequence of first-order phase transitions takes place when $h\to 0$ and the temperature is fixed.

KEY WORDS: SOS model; layering transition; entropic repulsion; cluster expansion; dominant ground states.

1. INTRODUCTION

The solid-on-solid (SOS) model is the simplest model of random surfaces and a large number of different effects can be studied in its framework. Reference 1 may serve as a general review of surface statistical physics and ref. 3 contains a detailed investigation of various SOS models.

In the most visualizable case $d=2$ our discrete two-dimensional surface lives in the upper half-space of $\mathbf{R}^d$. Due to thermal fluctuations this surface is repelled from the lower half-space as from a rigid wall, while an external magnetic field acts as an attractive potential. We prove that the competition between the entropic repulsion and the attractive potential results in the following behavior of the model. For sufficiently large inverse temperature β and for the magnetic field h of order $(1/\beta)e^{-4\beta k}$ the surface is localized at the level k. When the value of h decreases from $(1/\beta)e^{-4\beta k}$ to $(1/\beta)e^{-4\beta(k+1)}$ then at some point h_k^* two phases become stable in the thermodynamic limit. They are localized at levels k and $k+1$, respectively.

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Up to the first-order terms of the perturbation theory the value h_k^* was calculated in ref. 3, which also contains a transparent qualitative explanation of the layering transition phenomenon as well as a deep analysis of related questions.

The investigation of the layering transition in the semi-infinite Ising model seems to be the most natural generalization of our study and it will be the subject of a forthcoming paper. In fact, this result was announced in ref. 2 and then it was partially proven in ref. 7, but the full proof still is absent in the literature.

In a recent paper[5] the localization of the surface was established for the Gaussian SOS model but in different settings.

The SOS model with two-sided constraints was studied in ref. 11, the methods of which are close to our scheme, but the proof here is more direct and simple.

The next section contains an exact formulation of the result as well as its proof. Our approach is based on the cluster expansion technique,[8-10,13] on the Pirogov–Sinai theory,[12,14,15] and on the dominant ground-states theory.[4,6] In contrast with refs. 4, 6 and 11, we do not use so-called contour models with interaction. In the proof we concentrate mainly on the detailed construction of the cluster expansion for the logarithm of the partition function for the simplest case $d=2$ and $h \notin [h_k^* - \varepsilon, h_k^* + \varepsilon]$. This obviously proves the existence of the limit Gibbs measure for given values of parameters. After the cluster expansion is constructed the proof of the uniqueness of the corresponding Gibbs measure as well as the investigation of the vicinity of h_k^* become a standard application of the Pirogov–Sinai theory. So we give only a sketch of the corresponding proofs. The generalization on the case $d \geq 3$ seems to be straightforward and is omitted.

2. MODEL, RESULTS, AND PROOF

The model under consideration is defined by the Hamiltonian

$$H(\phi) = \tfrac{1}{2} \sum_{\langle x, x' \rangle} |\phi_x - \phi_{x'}| + h \sum_x \phi_x \tag{2.1}$$

where $\phi_x, \phi_{x'} \in \mathbf{Z}^+ = \{1, 2, ...\}$ are the spin variables at sites $x, x' \in \mathbf{Z}^d$; $h > 0$ is the external magnetic field; the first sum is taken over all nearest-neighbor pairs $\langle x, x' \rangle \in \mathbf{Z}^d$; and the second one extends over all sites $x \in \mathbf{Z}^d$.

For the sake of simplicity we fix $d=2$. The case $d>2$ can be treated in an analogous way. For any finite volume $V \subset \mathbf{Z}^2$ and for any fixed configuration ϕ'_{V^c} on its complement $V^c = \mathbf{Z}^2 \backslash V$ let

$$H(\phi_V \mid \phi'_{V^c}) = \tfrac{1}{2} \sum_{\langle x,x'\rangle \in V} |\phi_x - \phi_{x'}| + \tfrac{1}{2} \sum_{\langle x,x'\rangle : x\in V, x'\in V^c} |\phi_x - \phi'_{x'}| + h \sum_{x\in V} \phi_x \tag{2.2}$$

be a conditional Hamiltonian and let

$$\Xi(V \mid \phi'_{V^c}) = \sum_{\phi_V} \exp\{-\beta H(\phi_V \mid \phi'_{V^c})\} \tag{2.3}$$

be the corresponding partition function in the volume V with the boundary condition ϕ'_{V^c}. Here and below β is the inverse temperature and the notation ϕ_V is used for the configuration in the volume V, i.e., for the function ϕ_V: $V \mapsto \mathbf{Z}^+$. Denote by $\phi^{(k)}$ the configuration with $\phi_x^{(k)} = k$ for all $x \in \mathbf{Z}^2$. Our result is given by the following theorem.

Theorem 2.1. There exist a constant β_0 and a sequence of continuous functions $h_k^*(\beta)$, $k = 1, 2,...$ $[h_k^*(\beta) \searrow 0$ as $k \to \infty$ and β is fixed$]$, such that for any $\beta \geq \beta_0$ the following hold true:

(i) If $h_k^*(\beta) < h < h_{k-1}^*(\beta)$ $[h_0^*(\beta) \equiv +\infty]$, then the model possesses a unique $\mathbf{Z}^2$-periodic Gibbs state generated by the boundary condition $\phi^{(k)}$.

(ii) If $h = h_k^*(\beta)$, then the set of $\mathbf{Z}^2$-periodic extremal Gibbs states contains precisely two elements generated by the boundary conditions $\phi^{(k)}$ and $\phi^{(k+1)}$.

Proof. Every configuration ϕ of the SOS model can be naturally considered as a surface imbedded into $\mathbf{R}^3$. To get this surface we draw the horizontal (i.e., parallel to $\mathbf{Z}^2$) unit plaquettes centered at the points $(x, \phi_x) \in \mathbf{Z}^2 \times \mathbf{Z}^+$ and for every pair of nearest neighbors $\langle x, x'\rangle \in \mathbf{Z}^2$ we draw the sequence of $|\phi_x - \phi'_{x'}|$ stacked vertical (orthogonal to $\mathbf{Z}^2$) plaquettes centered at the points (y, a), $(y, a+1)$,..., (y, b), where $y = (x + x')/2$, $a = \min(\phi_x, \phi_{x'}) + \frac{1}{2}$, $b = \max(\phi_x, \phi_{x'}) - \frac{1}{2}$. Geometrically the surface ϕ consists of horizontal "ceilings" and vertical "walls."

Let a horizontal section of the surface ϕ be a subset of the vertical plaquettes of ϕ centered at the same horizontal plane. Every horizontal section of the surface ϕ can be uniquely decomposed onto connected components. A union of the components having the same projection on the underlying dual lattice $\tilde{\mathbf{Z}}^2 = \mathbf{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ we call a *cylinder* and we treat the surface ϕ as the collection of the cylinders.

Formally we define a cylinder as a triple $\gamma = (\tilde{\gamma}, E, I)$, where $\tilde{\gamma}(\gamma)$ is called the base of the cylinder, $E(\gamma)$ is called the external or starting level of the cylinder, and $I(\gamma)$ is called the internal or ending level of the cylinder. The base $\tilde{\gamma}$ is defined as a connected set of bonds of $\tilde{\mathbf{Z}}^2$ such that only an even number of bonds passes through every point $y \in \mathbf{Z}^2$; $E(\gamma)$ and $I(\gamma)$ are distinct positive integers. The value $S(\gamma) = \text{sign}(I(\gamma) - E(\gamma))$ is called the sign of the cylinder and $L(\gamma) = |I(\gamma) - E(\gamma)|$ is called the length of the cylinder. The interior of the cylinder $\bar{\gamma}(\gamma) = \bar{\gamma}(\tilde{\gamma}(\gamma))$ is the set of points $x \in \mathbf{Z}^2$ enclosed by $\tilde{\gamma}(\gamma)$. The configuration $\phi(\gamma)$ such that $\phi(\gamma)_x = E(\gamma)$, if $x \in \bar{\gamma}^c$ and $\phi(\gamma)_x = I(\gamma)$, if $x \in \bar{\gamma}$ naturally corresponds to the cylinder γ and the geometry of the surface $\phi(\gamma)$ justifies our notations and our terminology. Obviously $\tilde{\gamma}$ is the projection on $\mathbf{Z}^2$ of the vertical wall of ϕ, $E(\gamma)$ is the level of the ceiling which is adjacent to the vertical wall of γ from the exterior, and $I(\gamma)$ is the level of the ceiling which is adjacent to the vertical wall of γ from the interior. To establishing the one-to-one correspondence between the set of configurations and the set of collections of cylinders we introduce the *compatibility condition for cylinders.*

The cylinders $\gamma' = (\tilde{\gamma}', E', I')$ and $\gamma'' = (\tilde{\gamma}'', E'', I'')$ are *weakly compatible* if the following conditions are fulfilled:

(i) $S(\gamma') = S(\gamma'')$, $\bar{\gamma}' \cap \bar{\gamma}'' = \varnothing$ and $\tilde{\gamma}' \cap \tilde{\gamma}'' = \varnothing$; or $S(\gamma') = S(\gamma'')$ and $\bar{\gamma}' \subset \bar{\gamma}''$; or $S(\gamma') = S(\gamma'')$ and $\bar{\gamma}'' \subset \bar{\gamma}'$. (Here and further the sign $\subset$ denotes strict inclusion, in contrast with the sign $\subseteq$ of weak inclusion.)

(ii) $S(\gamma') = -S(\gamma'')$ and $\bar{\gamma}' \cap \bar{\gamma}'' = \varnothing$; or $S(\gamma') = -S(\gamma'')$, $\bar{\gamma}' \subset \bar{\gamma}''$, and $\tilde{\gamma}' \cap \tilde{\gamma}'' = \varnothing$; or $S(\gamma') = -S(\gamma'')$, $\bar{\gamma}'' \subset \bar{\gamma}'$, and $\tilde{\gamma}' \cap \tilde{\gamma}'' = \varnothing$.

The conditions (i)–(ii) differ slightly from the standard hard-cone condition $\bar{\gamma}' \cap \bar{\gamma}'' = \varnothing$ because we mean that $\tilde{\gamma}'$ and $\tilde{\gamma}''$ may have some number of common bonds, if it is not mentioned that $\tilde{\gamma}' \cap \tilde{\gamma}'' = \varnothing$. The cylinders γ' and γ'' are *compatible* if, in addition, the following phase matching condition are fulfilled:

(iii) $E(\gamma') = E(\gamma'')$ if $\bar{\gamma}' \cap \bar{\gamma}'' = \varnothing$; $E(\gamma') = I(\gamma'')$ if $\bar{\gamma}' \subset \bar{\gamma}''$; $I(\gamma') = E(\gamma'')$ if $\bar{\gamma}'' \subset \bar{\gamma}'$.

We say that cylinders γ' and γ'' are separated by the cylinder γ if $\bar{\gamma}' \subset \bar{\gamma} \subset \bar{\gamma}''$, or $\bar{\gamma}'' \subset \bar{\gamma} \subset \bar{\gamma}'$, or $\bar{\gamma}' \subset \bar{\gamma}$, $\bar{\gamma}'' \subset \bar{\gamma}^c$, or $\bar{\gamma}'' \subset \bar{\gamma}$, $\bar{\gamma}' \subset \bar{\gamma}^c$. A collection $\{\gamma_i\}$ is called a *compatible collection of cylinders* if any two its cylinders not separated by the third one are compatible.

To construct the configuration $\phi(\{\gamma_i\})$ uniquely corresponding to the finite compatible collection of cylinders we introduce a partial order for cylinders according to the partial order by inclusion for their interiors. Maximal cylinders we call external cylinders. For any finite compatible collection of cylinders $\{\gamma_i\}$ all external cylinders in this collection have the same external level $E(\{\gamma_i\})$. Let $\gamma(x)$ denote the minimal cylinder from $\{\gamma_i\}$ such that $\bar{\gamma}(x) \ni x$. We put $\phi_x(\{\gamma_i\}) = I(\gamma(x))$ if $\gamma(x) \neq \varnothing$ and $\phi_x(\{\gamma_i\}) = E(\{\gamma_i\})$ if $\gamma(x) = \varnothing$ (i.e. $\gamma(x)$ does not exist).

To any cylinder $\gamma = (\tilde{\gamma}, E, I)$ we assign the statistical weight

$$w(\gamma) = \exp\{-\beta L(\gamma)\,|\tilde{\gamma}| - \beta h S(\gamma)\, L(\gamma)\, |\bar{\gamma}|\} \tag{2.4}$$

where $|\tilde{\gamma}|$ denotes the number of bonds in $\tilde{\gamma}$, and $|\bar{\gamma}|$ denotes the number of sites in $\bar{\gamma}$. For the configuration ϕ such that $|\{x \in \mathbf{Z}^2: \phi_x \neq k\}| < \infty$ we obviously have

$$\exp\{-\beta H(\phi)\} = \exp\{-\beta H(\phi^{(k)})\} \prod_i w(\gamma_i(\phi)) \tag{2.5}$$

This shows the equivalence of the geometric language of cylinders with the original language of configurations. Both of them are used freely in the sequel.

The proof of Theorem 2.1 is based on the cluster expansion technique. As a first step the appropriate cluster representation for the partition function (2.3) is obtained below after a number of transformations. Some notations are needed. We write $\{\gamma_i\} \in V^{(k)}$ if $E(\gamma_i^{\text{ext}}) = k$ and $\bar{\gamma}_i^{\text{ext}} \subseteq V$ for every external cylinder γ_i^{ext} in the collection $\{\gamma_i\}$. We write $\{\gamma_i\} \in V^{(k,\pm)}$ if for $\gamma = (\partial V, k \mp 1, k)$ the extended collection $\{\gamma, \gamma_i\}$ remains compatible. Here $\partial V \in \tilde{\mathbf{Z}}^2$ denotes the boundary of V and the volume V supposed to be simply connected. With the help of the notations just introduced we rewrite $\Xi(V \mid \phi_{V^c}^{(k)})$ as

$$\Xi(V^{(k)}) = e^{-\beta hk\,|V|} \sum_{\{\gamma_i\} \in V^{(k)}} \prod_i w(\gamma_i) \tag{2.6}$$

and we also define slightly different partition functions

$$\Xi(V^{(k,\pm)}) = e^{-\beta hk\,|V|} \sum_{\{\gamma_i\} \in V^{(k,\pm)}} \prod_i w(\gamma_i) \tag{2.7}$$

The role of the last two partition functions becomes clear because of the following lemma.

Lemma 2.1. For any finite simply connected volume V and any $k \in \mathbf{Z}^+$

$$\Xi(V^{(k,\cdot)}) = \exp\{-\beta hk\,|V|\} \times \sum_{\{\gamma_i\}^{\text{ext}} \in V^{(k,\cdot)}} \prod_i w(\gamma_i) \exp\{\beta h I(\gamma_i)\,|\bar{\gamma}_i|\}\, \Xi(\bar{\gamma}_i^{(I(\gamma_i), S(\gamma_i))}) \tag{2.8}$$

where $V^{(k,\cdot)}$ denotes any of $V^{(k)}$, $V^{(k,+)}$ and $V^{(k,-)}$ and $\{\gamma_i\}^{\text{ext}}$ consists of mutually external cylinders only.

Proof. The proof is standard.[14] ∎

Another expression for $\Xi(V^{(k,\cdot)})$ can be obtained by means of a so-called "phase exchanging trick"[10,15] which is described below. We denote by $[\gamma_i]$ a *weakly compatible collection of cylinders*, i.e., a collection for which any two contours not separated by a third one are weakly compatible. We write $[\gamma_i] \in V^{(k)}$ if $E(\gamma_i) = k$, $\bar{\gamma}_i \subseteq V$ for all cylinders of $[\gamma_i]$. Analogously $[\gamma_i] \in V^{(k,\pm)}$ means that $E(\gamma_i) = k$ for all cylinders of $[\gamma_i]$ and the extended collection $[\gamma, \gamma_i]$ with $\gamma = (\partial V, k \mp 1, k)$ is still weakly compatible.

Lemma 2.2. Let the renormalized statistical weight of $\gamma = (\bar{\gamma}, E, I)$ be equal to

$$\tilde{w}(\gamma) = \exp\{-\beta L\,|\bar{\gamma}|\}\,\frac{\Xi(\bar{\gamma}^{(I,S)})}{\Xi(\bar{\gamma}^{(E,S)})} \tag{2.9}$$

where $S = S(\gamma)$, $L = L(\gamma)$. Then for any simply connected volume V and any $k \in \mathbf{Z}^+$

$$\Xi(V^{(k,\cdot)}) = e^{-\beta hk\,|V|} \sum_{[\gamma_i] \in V^{(k,\cdot)}} \prod_i \tilde{w}(\gamma_i) \tag{2.10}$$

Proof. For a unit volume Lemma 2.2 is obvious. Now we proceed by the induction on V. Suppose that Lemma 2.2 is verified for all $V' \subset V$. Then Lemma 2.1 gives us

$$\begin{aligned}
\Xi(V^{(k,\cdot)}) &= \exp\{-\beta hk\,|V|\} \\
&\quad\times \sum_{\{\gamma_i\}^{\text{ext}} \in V^{(k,\cdot)}} \prod_i w(\gamma_i) \exp\{\beta h I(\gamma_i)\,|\bar{\gamma}_i|\}\, \Xi(\bar{\gamma}_i^{(I(\gamma_i), S(\gamma_i))}) \\
&= \exp\{-\beta hk\,|V|\} \\
&\quad\times \sum_{\{\gamma_i\}^{\text{ext}} \in V^{(k,\cdot)}} \prod_i \left[w(\gamma_i)\, \frac{\exp\{\beta h I(\gamma_i)\,|\bar{\gamma}_i|\}\, \Xi(\bar{\gamma}_i^{(I(\gamma_i), S(\gamma_i))})}{\exp\{\beta hk\,|\bar{\gamma}_i|\}\, \Xi(\bar{\gamma}_i^{(k, S(\gamma_i))})} \right] \\
&\quad\times [\exp\{\beta hk\,|\bar{\gamma}_i|\}\, \Xi(\bar{\gamma}_i^{(k, S(\gamma_i))})]
\end{aligned}$$

The expression inside the first brackets is equal to $\tilde{w}(\gamma_i)$ and the expression inside the second brackets is equal to

$$\sum_{[\gamma_j] \in \bar{\gamma}_i^{(k, S(\gamma_i))}} \prod_j \tilde{w}(\gamma_j)$$

due to the induction assumption. Therefore

$$\begin{aligned}
\Xi(V^{(k,\cdot)}) &= e^{-\beta hk\,|V|} \sum_{\{\gamma_i\}^{\text{ext}} \in V^{(k,\cdot)}} \prod_i \left[\tilde{w}(\gamma_i) \sum_{[\gamma_j] \in \bar{\gamma}_i^{(k, S(\gamma_i))}} \prod_j \tilde{w}(\gamma_j) \right] \\
&= e^{-\beta hk\,|V|} \sum_{[\gamma_i] \in V^{(k,\cdot)}} \prod_i \tilde{w}(\gamma_i) \quad \blacksquare
\end{aligned}$$

Both representations (2.6)–(2.7) and (2.10) for the partition functions $\Xi(V^{(k,\cdot)})$ have their pro and contra. The first one uses more controllable statistical weights $w(\cdot)$ but requires a more complicated (and, in fact, nonlocal) compatibility condition. On the contrary, the second representation uses a weak compatibility condition of hard-core type but the renormalized statistical weights are less controllable. Our strategy is to find an optimal balance between (2.6)–(2.7) and (2.10). For this purpose we introduce the concept of an *elementary cylinder*. Generally speaking, we divide the class of all cylinders into two subclasses. The cylinders from the first subclass we call elementary cylinders and use special notations $\varepsilon = (\tilde{\varepsilon}, E, I)$, $\bar{\varepsilon}(\varepsilon)$, $L(\varepsilon)$, $S(\varepsilon)$ for them. The cylinders from the second subclass we call nonelementary cylinders and keep the notations $\gamma = (\tilde{\gamma}, E, I)$, $L(\gamma)$, $S(\gamma)$ for them only. The word "nonelementary" is usually omitted if it cannot lead to any misunderstanding. The notation $\{\gamma_i; \varepsilon_j\}$ is used for the compatible collection of cylinders and elementary cylinders; the notation $[\gamma_i; \varepsilon_j]$ is used for the weakly compatible collection of cylinders and elementary cylinders. The mixed notation $\{\gamma_i; \varepsilon_j]$ is introduced for the collection of cylinders and elementary cylinders with the following compatibility properties:

(i) All of the collection is weakly compatible.

(ii) For any pair of nonelementary cylinders $\gamma_{i'}$ and $\gamma_{i''}$ not separated by a third nonelementary cylinder, $E_{i'} = E_{i''}$ if $\bar{\gamma}_{i'} \cap \bar{\gamma}_{i''} = \varnothing$, $E_{i'} = I_{i''}$ if $\bar{\gamma}_{i'} \subset \bar{\gamma}_{i''}$, and $I_{i'} = E_{i''}$ if $\bar{\gamma}_{i''} \subset \bar{\gamma}_{i'}$.

(iii) For any pair of elementary cylinders $\varepsilon_{j'}$ and $\varepsilon_{j''}$ not separated by a nonelementary cylinder $E_{j'} = E_{j''}$.

(iv) For any pair of elementary cylinder ε_j and nonelementary cylinder γ_i not separated by another nonelementary cylinder $E_j = E_i$ if $\bar{\varepsilon}_j \cap \bar{\gamma}_i = \varnothing$, $E_j = I_i$ if $\bar{\varepsilon}_j \subset \bar{\gamma}_i$, and $E_j = E_i$ (not $E_j = I_i$!) if $\bar{\gamma}_i \subset \bar{\varepsilon}_j$.

To explain the notation $\{\cdot;\cdot]$ we note that the nonelementary part of this collection is compatible, the elementary one is weakly compatible, and some mixed compatibility condition is fulfilled between its elementary and nonelementary parts. For every collection $\{\gamma_i; \varepsilon_j]$ we put in correspondence the configuration $\phi(\{\gamma_i; \varepsilon_j])$ such that $\phi_x(\{\gamma_i; \varepsilon_j]) = I(\gamma(x))$ if $\gamma(x)$ is a nonelementary cylinder, $\phi_x(\{\gamma_i; \varepsilon_j]) = E(\gamma(x))$ if $\gamma(x)$ is an elementary cylinder, and $\phi_x(\{\gamma_i; \varepsilon_j]) = E(\{\gamma_i; \varepsilon_j])$ if $\gamma(x) = \varnothing$. Let us stress that $\phi(\{\gamma_i; \varepsilon_j]) \neq \phi(\{\gamma_i; \varepsilon_j\})$ because we mean that according to the "phase exchanging trick" the elementary cylinder ε_j does not change the configuration from $E(\varepsilon_j)$ to $I(\varepsilon_j)$. This leads us to the following representation for the $\Xi(V^{(k,\cdot)})$.

Lemma 2.3. For any finite simply connected volume V and any $k \in \mathbf{Z}^+$

$$\Xi(V^{(k,\cdot)}) = e^{-\beta h k |V|} \sum_{\{\gamma_i; \varepsilon_j] \in V^{(k,\cdot)}} \prod_i w(\gamma_i) \prod_j \tilde{w}(\varepsilon_j) \tag{2.11}$$

Proof. For unit V, Lemma 2.3 is obvious. Suppose that it is proven for all $V' \subset V$. Then by Lemma 2.1

$$\begin{aligned}
\Xi(V^{(k,\cdot)}) &= \exp\{-\beta hk\,|V|\} \\
&\quad\times \sum_{\{\gamma_i;\varepsilon_j\}^{\text{ext}} \in V^{(k,\cdot)}} \left\{\prod_i w(\gamma_i) \exp\{\beta h I(\gamma_i)\,|\bar\gamma_i|\}\, \Xi(\bar\gamma_i^{(I(\gamma_i),S(\gamma_i))})\right\} \\
&\quad\times \left\{\prod_j w(\varepsilon_j) \exp\{\beta h I(\varepsilon_j)\,|\bar\varepsilon_j|\}\, \Xi(\bar\varepsilon_j^{(I(\varepsilon_j),S(\varepsilon_j))})\right\} \\
&= \exp\{-\beta hk\,|V|\} \\
&\quad\times \sum_{\{\gamma_i;\varepsilon_j\}^{\text{ext}} \in V^{(k,\cdot)}} \left\{\prod_i w(\gamma_i) \exp\{\beta h I(\gamma_i)\,|\bar\gamma_i|\}\, \Xi(\bar\gamma_i^{(I(\gamma_i),S(\gamma_i))})\right\} \\
&\quad\times \left\{\prod_j \tilde w(\varepsilon_j) \exp\{\beta h E(\varepsilon_j)\,|\bar\varepsilon_j|\}\, \Xi(\bar\varepsilon_i^{(E(\varepsilon_j),S(\varepsilon_j))})\right\}
\end{aligned}$$

Applying the induction hypothesis to the expressions in braces, we obtain

$$\begin{aligned}
\Xi(V^{(k,\cdot)}) &= e^{-\beta hk\,|V|} \sum_{\{\gamma_i;\varepsilon_j\}^{\text{ext}} \in V^{(k,\cdot)}} \left\{\prod_i w(\gamma_i)\right. \\
&\quad\times \left.\left[\sum_{\{\gamma_m;\varepsilon_n\} \in \bar\gamma_i^{(I(\gamma_i),S(\gamma_i))}} \prod_m w(\gamma_m) \prod_n \tilde w(\varepsilon_n)\right]\right\} \\
&\quad\times \left\{\prod_j \tilde w(\varepsilon_j) \left[\sum_{\{\gamma_p;\varepsilon_q\} \in \bar\varepsilon_j^{(E(\varepsilon_j),S(\varepsilon_j))}} \prod_p w(\gamma_p) \prod_q \tilde w(\varepsilon_q)\right]\right\} \\
&= e^{-\beta hk\,|V|} \sum_{\{\gamma_t;\varepsilon_u\} \in V^{(k,\cdot)}} \prod_t w(\gamma_t) \prod_u \tilde w(\varepsilon_u) \qquad \blacksquare
\end{aligned}$$

Besides the partition function $\Xi(V^{(k,\cdot)})$, we also use the partition function

$$Z(V^{(k,\cdot)}) = \sum_{[\varepsilon_j] \in V^{(k,\cdot)}} \prod_j \tilde w(\varepsilon_j) \tag{2.12}$$

It is the sum over weakly compatible collections of elementary cylinders peculiar to the phase $\phi^{(k)}$ only. So we refer to $Z(V^{(k,\cdot)})$ as the partition function over the gas of the elementary cylinders of the surface $\phi^{(k)}$ (restricted ensemble in the terminology of ref. 4).

Our last transformation of $\Xi(V^{(k,\cdot)})$ consist of some grouping of γ_i. For this purpose we fix some $k \in \mathbf{Z}^+$ and we define a contour $\Gamma =$

$\{\gamma^{\text{ext}}, \gamma_i, \gamma^{\text{int},j}\}$ as the compatible collection of nonelementary cylinders with the following properties:

(i) $\bar{\gamma}^{\text{ext}}(\Gamma)$ contains the interiors of all other cylinders of Γ and $E(\gamma^{\text{ext}}(\Gamma)) = k$, while $E(\gamma_j) \neq k$, $E(\gamma^{\text{int},j}) \neq k$ for all i, j.

(ii) For every j the interior $\bar{\gamma}^{\text{int},j}$ of the cylinder $\gamma^{\text{int},j}$ does not contain the interior of any other cylinder of Γ and $I(\gamma^{\text{int},j}) = k$, while $I(\gamma^{\text{ext}}) \neq k$ as well as $I(\gamma_i) \neq k$ for all i.

In other words the cylinder $\gamma^{\text{ext}}(\Gamma)$ is maximally external and the cylinders $\gamma^{\text{int},j}(\Gamma)$ are maximally internal in the collection Γ. The volume $\text{Supp}(\Gamma) = \bar{\gamma}^{\text{ext}}(\Gamma) \setminus (\bigcup_j \bar{\gamma}^{\text{int},j}(\Gamma))$ is called the support of Γ. To every cylinder γ_i there corresponds a set

$$\text{Supp}_i(\Gamma) = \bigcap_{\gamma \in \Gamma, \bar{\gamma} \subset \bar{\gamma}_i} (\bar{\gamma}_i \setminus \bar{\gamma})$$

and analogously we define a set

$$\text{Supp}_e(\Gamma) = \bigcap_{\gamma \in \Gamma, \gamma \neq \gamma^{\text{ext}}} (\bar{\gamma}^{\text{ext}} \setminus \bar{\gamma})$$

which corresponds to γ^{ext}. These sets are called the components of the support of Γ because they are mutually disjoint and their union coincides with $\text{Supp}(\Gamma)$. It is important that the surface $\phi(\Gamma)$ uniquely defined by Γ is constant on every $\text{Supp}_i(\Gamma)$, i.e., $\phi(\Gamma)$ can be described as

$$\begin{aligned} \phi_x(\Gamma) &= I(\gamma_i) && \text{if} \quad x \in \text{Supp}_i(\Gamma) \\ \phi_x(\Gamma) &= I(\gamma^{\text{ext}}) && \text{if} \quad x \in \text{Supp}_e(\Gamma) \\ \phi_x(\Gamma) &= E(\gamma^{\text{ext}}) && \text{if} \quad x \notin \text{Supp}(\Gamma) \end{aligned}$$

This shows that Γ is the object peculiar to the phase $\phi^{(k)}$ and we put $E(\Gamma) = E(\gamma^{\text{ext}}) = I(\gamma^{\text{int},j}) = k$. An arbitrary collection $\{\gamma_m; \varepsilon_n] \in V^{(k,\cdot)}$ uniquely defines the configuration $\phi(\{\gamma_m; \varepsilon_n])$. If we consider the connected components of the set $\{x \in \mathbf{Z}^2 : \phi_x(\{\gamma_m; \varepsilon_n]) \neq k\}$, then it can be easily checked that every such component is the support of some contour Γ. This property justifies the definition of Γ.

Two contours are called *compatible* if their supports are nonintersecting. The collection of contours $\{\Gamma_l\}$ is called a *compatible collection of contours* if any two contours in it are compatible. In terms of compatible collections of contours the partition function $\Xi(V^{(k,\cdot)})$ can be represented as

$$\Xi(V^{(k,\cdot)}) = e^{-\beta hk|V|} \sum_{\{\gamma_i;\varepsilon_j\} \in V^{(k,\cdot)}} \prod_i w(\gamma_i) \prod_j \tilde{w}(\varepsilon_j)$$

$$= e^{-\beta hk|V|} \sum_{\{\Gamma_l\} \in V^{(k,\cdot)}} Z\left(\left(V \Big\backslash \left(\bigcup_l \operatorname{Supp}(\Gamma_l)\right)\right)^{(k,\dots)}\right)$$

$$\times \left\{ \prod_l \left[\prod_j w(\gamma^{\text{int},j}(\Gamma_l)) \right] \right.$$

$$\times \left[w(\gamma^{\text{ext}}(\Gamma_l))\, Z(\operatorname{Supp}_e(\Gamma_l)^{(I(\gamma^{\text{ext}}(\Gamma_l)),\dots)}) \right]$$

$$\left. \times \left[\prod_i w(\gamma_i(\Gamma_l))\, Z(\operatorname{Supp}_i(\Gamma_l)^{(I(\gamma_i(\Gamma_l)),\dots)}) \right] \right\} \tag{2.13}$$

where the notation $V^{(m,\dots)}$ is used for the not simply connected volume with the boundary condition $\phi^{(m)}$ specified on V^c and with some not mentioned but clear from the context sign ($+$ or $-$) specified on the connected components of ∂V. Define the statistical weight of the contour $\Gamma = \{\gamma^{\text{ext}}, \gamma_i, \gamma^{\text{int},j}\}$ as

$$w(\Gamma) = Z^{-1}(\operatorname{Supp}(\Gamma)^{(E(\Gamma),\dots)})$$

$$\times \left[\prod_j w(\gamma^{\text{int},j}) \right] \left[w(\gamma^{\text{ext}})\, Z(\operatorname{Supp}_e(\Gamma)^{(I(\gamma^{\text{ext}}),\dots)}) \right]$$

$$\times \left[\prod_i w(\gamma_i)\, Z(\operatorname{Supp}_i(\Gamma)^{(I(\gamma_i),\dots)}) \right] \tag{2.14}$$

Our final expression for $\Xi(V^{(k,\cdot)})$ is the following.

Lemma 2.4. For any finite simply connected volume V and fixed $k \in \mathbf{Z}^+$

$$\Xi(V^{(k,\cdot)}) = e^{-\beta hk|V|} \sum_{\{\Gamma_l\} \in V^{(k,\cdot)}} Z\left(\left(V \Big\backslash \left(\bigcup_l \operatorname{Supp}(\Gamma_l)\right)\right)^{(k,\dots)}\right)$$

$$\times \prod_l w(\Gamma_l)\, Z(\operatorname{Supp}(\Gamma_l)^{(k,\dots)}) \tag{2.15}$$

Proof. If we substitute (2.14) into (2.13), then we obtain (2.15). ∎

Expression (2.15) is a typical cluster representation of the partition function $\Xi(V^{(k,\cdot)})$.[10] Indeed, our clusters are objects of twofold origin: contours and elementary cylinders. Both of them are labeled by the phase $\phi^{(k)}$: the external levels of elementary cylinders and the external levels of contours are equal to k. Clusters interact only by means of the hard-core compatibility condition: supports of contours are mutually disjoint, bases of elementary cylinders are mutually disjoint and do not intersect with the boundaries of the supports of contours. [Clearly, $\partial\,\mathrm{Supp}(\Gamma) = \bigcup_j \tilde{\gamma}^{\mathrm{int},j} \cup \tilde{\gamma}^{\mathrm{ext}}$.] To be precise we should mention that elementary cylinders and boundaries of supports of contours are weakly compatible, which means that sometimes they may touch each other. To pass from the cluster representation of the partition function to the cluster expansion of its logarithm we need "good" estimates for the statistical weights $w(\Gamma)$ and $\tilde{w}(\varepsilon)$. Up to now our calculations were quite general and, in fact, not specific to the model under consideration. The construction of appropriate estimations of statistical weights is of course most important and model dependent. Here we follow the strategy of the dominant ground-states theory,[4,6] but our proofs are self-contained. First we prove the weak version of part (i) of Theorem 2.1 to demonstrate our technique on the simplest example.

Theorem 2.2. There exists a constant $\beta_0 > 0$ such that for $\beta \geqslant \beta_0$ and

$$\frac{1}{\beta}\exp\left\{-4\beta k + \frac{\beta}{100}\right\} \leqslant h \leqslant \frac{1}{\beta}\exp\left\{-4\beta(k-1) - \frac{\beta}{100}\right\}$$

the model possesses a unique $\mathbf{Z}^2$-periodic Gibbs state generated by the boundary condition $\phi^{(k)}$.

Here and below we use the number 100 when we need some large absolute constant. We do not mean that in all cases it necessarily should be the same constant or that it is the optimal one. This fixed choice produces other absolute constants and we often write them as $2\cdot 100$, $4/100$, etc., to show the origin of such constants.

Proof. Now we specify the notions of an elementary cylinder and nonelementary cylinder. A cylinder $\gamma = (\tilde{\gamma}, E, I)$ is called a *nonelementary cylinder*, if

$$\operatorname{diam}\tilde{\gamma} > 100 \min\left(E, -\left[\frac{1}{4\beta}\ln(\beta h)\right]\right) \tag{2.16}$$

where $[\cdot]$ denotes the integral part. The condition above is geometric, in contrast with the energetic condition used in refs. 4 and 6. It means that we have no fixed order in powers of $e^{-\beta}$ as the borderline between the statistical weights of elementary and nonelementary cylinders. This is important for the lemma below, which plays a key role in our proof. In it we estimate $\tilde{w}(\varepsilon)$ and quantitatively describe the entropic repulsion phenomenon:

Lemma 2.5. For any elementary cylinder ε define the statistical weights

$$w_0(\varepsilon) = \exp\{-\beta\, |\tilde{\varepsilon}|\, L(\varepsilon)\}, \qquad w_1(\varepsilon) = \tilde{w}(\varepsilon) - w_0(\varepsilon) \tag{2.17}$$

Then for any $\beta \geqslant \beta_0$ and $k = -[(1/4\beta)\ln(\beta h)] \geqslant 1$ we have

(i) $$\min(\tilde{w}(\varepsilon), |w_1(\varepsilon)|) \leqslant \exp\left\{-\frac{\beta}{100}\, |\tilde{\varepsilon}|\, L(\varepsilon) - 4\beta \min(E(\varepsilon), k)\right\} \tag{2.18}$$

(ii) For any $0 < m < n$

$$\exp\left\{-2\,|V|\exp(-4\beta m) - |V|\exp\left(-4\beta\min(m,k) - \frac{\beta}{100}\right)\right\}$$
$$\leqslant \frac{Z(V^{(m,\cdot)})}{Z(V^{(n,\cdot)})}$$
$$\leqslant \exp\left\{-\frac{1}{2}\,|V|\exp(-4\beta m) + |V|\exp\left(-4\beta\min(m,k) - \frac{\beta}{100}\right)\right\} \tag{2.19}$$

Remark 1. Statements (i) and (ii) are united in one lemma because we prove them simultaneously. We need (i) to compare $Z(V^{(m,\cdot)})$ for different m and we need (ii) to estimate the statistical weight $\tilde{w}(\varepsilon)$.

Remark 2. The statistical weight $\tilde{w}(\varepsilon)$ of the elementary cylinder ε is the renormalized one. So $\tilde{w}(\varepsilon) \neq \tilde{w}(\varepsilon')$ for $\varepsilon = (\tilde{\varepsilon}, m, I)$ and $\varepsilon' = (\tilde{\varepsilon}, n, I + n - m)$. It makes the comparison of $Z(V^{(m,\cdot)})$ and $Z(V^{(n,\cdot)})$ more complicated. To handle this difficulty we split $\tilde{w}(\varepsilon)$ into the sum $\tilde{w}(\varepsilon) = w_0(\varepsilon) + w_1(\varepsilon)$ such that $w_0(\varepsilon)$ is invariant under vertical shifts and represents the main part of $\tilde{w}(\varepsilon)$ while $w_1(\varepsilon)$ is small.

Remark 3. If h satisfies the condition

$$\frac{1}{\beta}\exp\left\{-4\beta k + \frac{\beta}{100}\right\} \leqslant h \leqslant \frac{1}{\beta}\exp\left\{-4\beta(k-1) - \frac{\beta}{100}\right\}$$

of Theorem 2.2, then it follows from (2.19) that for $m \neq k$

$$\frac{\exp(-\beta h m\,|V|)\, Z(V^{(m,\cdot)})}{\exp(-\beta h k\,|V|)\, Z(V^{(k,\cdot)})} \leqslant \exp\left\{-\frac{1}{3}\,|V|\exp(-4\beta\min(m,k))\right\}$$

what means that $\phi^{(k)}$ is a unique dominant ground state for these values of h.

Proof. For unit ε (i.e, for ε with $|\varepsilon|=1$, $|\tilde{\varepsilon}|=4$) we have

$$\tilde{w}(\varepsilon)=e^{-4\beta L(\varepsilon)-\beta hS(\varepsilon)\,L(\varepsilon)}$$

Therefore, if $L(\varepsilon)\leqslant 100k$, then

$$|w_1(\varepsilon)|\leqslant e^{-4\beta L(\varepsilon)}2\beta hL(\varepsilon)\leqslant e^{-4\beta L(\varepsilon)}2L(\varepsilon)\,e^{-4\beta(k-1)-\beta/100}$$
$$\leqslant \exp\left\{-\frac{\beta}{100}|\tilde{\varepsilon}|\,L(\varepsilon)-4\beta\min(E(\varepsilon),k)\right\}$$

for β large enough. Most of the bounds which we obtain below are valid only for β larger than some absolute constant. So we state it here and do not mention it later. If $L(\varepsilon)>100k$, then

$$\tilde{w}(\varepsilon)\leqslant \exp\{-2\beta L(\varepsilon)\}\leqslant \exp\left\{-\frac{\beta}{100}|\tilde{\varepsilon}|\,L(\varepsilon)-4\beta\min(E(\varepsilon),k)\right\}$$

It is clear that $Z(V^{(m,\pm)})=1$ if $|V|=1$ and only unit elementary cylinders contribute to $Z(V^{(m)})$ with $|V|=1$ as well as to $Z(V^{(m,\pm)})$ with $|V|=2$. Hence

$$Z(V^{(m)})=1+\sum_{r=1}^{\infty}e^{-4\beta r-\beta hr}+\sum_{r=1}^{m-1}e^{-4\beta r+\beta hr} \tag{2.20}$$

where the first sum corresponds to the upward spikes (unit elementary cylinders) and the second one corresponds to the downward spikes. The entropic repulsion reveals itself in the fact that the number of downward spikes depends on m. For $V^{(m)}$ with $|V|=1$ the estimation (ii) easily follows from (2.20) and for $V^{(m,\pm)}$ with $|V|=2$ it can be obtained in an analogous way.

For larger volumes V the unit spikes also bring the main contribution to $Z(V^{(m,\cdot)})/Z(V^{(n,\cdot)})$ and to show this we apply the induction on the volume. Our aim is to reproduce (ii) for $V^{(m,\cdot)}$ supposing that (i) is proven for all $\varepsilon\in V^{(m,\cdot)}$. We extend the ensemble of elementary cylinders introducing two copies of ε for every ε with

$$|w_1(\varepsilon)|\leqslant \exp\left\{-\frac{\beta}{100}|\tilde{\varepsilon}|\,L(\varepsilon)-4\beta\min(E(\varepsilon),k)\right\}$$

We consider these two copies of ε as not weakly compatible with each other and assign the statistical weight $w_0(\varepsilon)$ for the first copy and the statistical weight $w_1(\varepsilon)$ for the second one. To avoid confusion, we denote by ρ the elementary cylinders of the extended ensemble and we denote by

$w(\rho)$ their statistical weights. This $w(\rho)$ is equal to $\tilde{w}(\varepsilon)$ if ρ corresponds to a nonsplit ε and it is equal to $w_0(\varepsilon)$ or $w_1(\varepsilon)$ if ρ corresponds to the first copy or the second copy of the split ε, respectively. Obviously we have the representation

$$Z(V^{(m,\cdot)}) = \sum_{[\rho_i] \in V^{(m,\cdot)}} \prod_i w(\rho_i) \tag{2.21}$$

with the bound

$$|w(\rho)| \leqslant \exp\left\{-\frac{\beta}{100}|\tilde{\rho}|\, L(\rho)\right\}$$

for the statistical weight of ρ. This bound allows us to write down an absolutely convergent cluster expansion for the logarithm of the partition function (2.21).

$$\ln Z(V^{(m,\cdot)}) = \sum_{\pi \in V^{(m,\cdot)}} r(\pi)\, w(\pi) \tag{2.22}$$

where the sum is extended over so-called polymers π belonging to the volume $V^{(m,\cdot)}$. Every polymer $\pi = (\rho_i^{\alpha_i})$ is a collection of elementary cylinders ρ_i with α_i copies of every ρ_i such that for every pair $\rho_{i'}$ and $\rho_{i''}$ there exists a sequence $\rho_{i'} = \rho_{i_1},..., p_{i_l} = \rho_{i''}$ with ρ_{i_j} and $\rho_{i_{j+1}}$ being not weakly compatible $(j = 1, 2,..., l-1)$. The statistical weight of π is defined as

$$w(\pi) = \prod_i w(\rho_i)^{\alpha_i}$$

The Möbius factor $r(\pi)$ is equal to

$$r(\pi) = \left(\prod_i \frac{1}{\alpha_i!}\right) \sum_{G(\pi)} (-1)^{|G(\pi)|} \tag{2.23}$$

where the sum is taken over all connected graphs $G(\pi)$ on the set of $\sum_i \alpha_i$ vertices labeled by ρ_i with the following property. If two vertices of $G(\pi)$ are joined by the edge, then the corresponding elementary cylinders are not weakly compatible. $|G(\pi)|$ is the number of edges in $G(\pi)$.[9,10,13] It is well known[13] that

$$|r(\pi)| \leqslant \exp\{c\, |\tilde{\pi}|\} \tag{2.24}$$

where $|\tilde{\pi}| = \sum_i \alpha_i\, |\tilde{\rho}_i|$ and c is a constant depending on the dimension. To calculate $\Delta = \ln Z(V^{(m,\cdot)}) - \ln Z(V^{(n,\cdot)})$ note the following.

For π such that it contains at least one elementary cylinder ρ with

$$w(\rho) \neq \exp\{-\beta L(\rho)\, |\tilde{\rho}|\}$$

we have the bound

$$|r(\pi)\, w(\pi)| \leqslant \exp\left\{-\frac{1}{2}\frac{\beta}{100}|\tilde{\pi}| - 4\beta \min(m, n, k)\right\} \tag{2.25}$$

because for these ρ

$$|w(\rho)| \leqslant \exp\left\{-\frac{\beta}{100}|\tilde{\rho}|\, L(\rho) - 4\beta \min(E(\rho), k)\right\}$$

The number of polymers π with fixed $|\tilde{\pi}|$ is greater than $\exp(c\,|\tilde{\pi}|)$, hence the total contribution to Δ coming from the polymers π satisfying the bound (2.25) is less than $\frac{1}{4}|V| \exp\{-4\beta \min(m, n, k) - \beta/100\}$.

The polymer π with

$$w(\pi) = \exp\left\{-\beta \sum_i \alpha_i L(\rho_i)\, |\tilde{\rho}_i|\right\}$$

is, of course, invariant under vertical shifts. This allows us to cancel a large number of common terms in $\ln Z(V^{(m,\cdot)})$ and $\ln Z(V^{(n,\cdot)})$. The polymers which are not canceled contain at least one ρ_0 with $L(\rho_0) \geqslant \min(m, n) = m$. If the number of the elementary cylinders in π is greater than 1, then the total contribution to Δ given by these π is less than $\frac{1}{4}|V|$ $\exp\{-4\beta \min(m, n, k) - \beta/100\}$.

Indeed, fix some elementary cylinder ρ_0 with $L(\rho_0) \geqslant m$ and consider an arbitrary polymer $\pi \ni \rho_0$. If we delete ρ_0 from π, then π is desintegrated into connected components π_j such that $\pi = (\bigcup_j \pi_j) \cup \rho_0$, $\tilde{\pi}_{j_1} \cap \tilde{\pi}_{j_2} = \varnothing$, $\tilde{\pi}_j \cap \tilde{\rho}_0 \neq \varnothing$. Denote by Δ_1 the sum of $|r(\pi)\, \omega(\pi)|$ taken over all π which contain at least two elementary cylinders with at least one of them having the length $L \geqslant m$. Then

$$\begin{aligned}
\Delta_1 &\leqslant |V| \sum_{\rho_0:\, \tilde{\rho}_0 \ni 0,\, L(\rho_0) \geqslant m} \ \sum_{\pi:\, \pi \ni \rho_0} |r(\pi)\, w(\pi)| \\
&\leqslant |V| \sum_{\rho_0:\, \tilde{\rho}_0 \ni 0,\, L(\rho_0) \geqslant m} w(\rho_0) \sum_{(\pi_j):\, \pi = (\bigcup_j \pi_j) \cup \rho_0} |r(\pi)| \prod_j w(\pi_j) \\
&\leqslant |V| \sum_{\rho_0:\, \tilde{\rho}_0 \ni 0,\, L(\rho_0) \geqslant m} w(\rho_0) \exp(c\,|\tilde{\rho}_0|) \sum_{(\pi_j):\, \pi = (\bigcup_j \pi_j) \cup \rho_0} \prod_j w(\pi_j) \exp(c\,|\tilde{\pi}_j|) \\
&\leqslant |V| \sum_{\rho_0:\, \tilde{\rho}_0 \ni 0,\, L(\rho_0) \geqslant m} w(\rho_0) \exp(c\,|\tilde{\rho}_0| - \beta) \\
&\quad \times \prod_{x \in \tilde{\rho}_0} \left(1 + \sum_{\tilde{\pi} \ni x} w(\pi) \exp(c\,|\tilde{\pi}| + \beta)\right) \\
&\leqslant |V| \sum_{\rho_0:\, \tilde{\rho}_0 \ni 0,\, L(\rho_0) \geqslant m} \\
&\quad \times \exp\{-\beta L(\rho_0)\, |\tilde{\rho}_0| + c\,|\tilde{\rho}_0| - \beta + |\tilde{\rho}_0| \exp(-\beta)\}
\end{aligned} \tag{2.26}$$

If $4 \leqslant |\tilde{\rho}_0| \leqslant 100$, then the RHS of (2.26) is less than

$$|V| \exp\{(2c + \exp(-\beta))\, 100 - \beta\} \sum_{l=m}^{\infty} \exp(-\beta 4l)$$

$$\leqslant |V| \exp\{(2c + \exp(-\beta))\, 100 - \beta\}\, 2 \exp(-4\beta m)$$

$$\leqslant \frac{1}{4} |V| \exp\left(-4\beta m - \frac{\beta}{100}\right)$$

If $|\tilde{\rho}_0| > 100$, then the RHS of (2.26) does not exceed

$$|V| \sum_{|\tilde{\rho}_0| \geqslant 100,\, L(\rho_0) \geqslant m} \exp\{-|\tilde{\rho}_0|\, (2c + \exp(-\beta) - \beta L(\rho_0))\}$$

$$\leqslant |V| \sum_{l=m}^{\infty} \exp\left(-100\beta \frac{l}{2}\right) \leqslant \frac{1}{4} |V| \exp\left(-4\beta m - \frac{\beta}{100}\right)$$

Clearly the contribution to Δ given by the polymers $\pi = (\rho)$ with $|\tilde{\rho}| > 4$ and $L(\rho) \geqslant m$ is less than $\frac{1}{4} |V| \exp\{-4\beta \min(m, k) - \beta/100\}$.

Thus an essential contribution to the difference Δ is given only by polymers $\pi = (\rho)$ with ρ being the unit spike. For these π obviously $r(\pi) = 1$, $w(\pi) = w(\rho) = e^{-4\beta L(\rho)}$. So we finally obtain

$$\Delta = -|V| \sum_{l=m}^{n-1} e^{-4\beta l} + \delta \tag{2.27}$$

where $|\delta| \leqslant 4\frac{1}{4} |V| \exp\{-4\beta \min(m, k) - \beta/100\}$. This implies (ii).

Let us stress that the difference between $V^{(\cdot)}$, $V^{(\cdot, +)}$, and $V^{(\cdot, -)}$ reveals itself in the following. If some elementary cylinder from some polymer π touches ∂V, then that π does not contribute to $\ln Z(V^{(\cdot, -S(\rho))})$, while it contributes to $\ln Z(V^{(\cdot)})$ and $\ln Z(V^{(\cdot, S(\rho))})$.

Now we come to the second part of the induction step. Here our aim is to reproduce (i) for $\varepsilon = (\tilde{\varepsilon}, E, I)$ supposing that (ii) is proven for all $V \subset \tilde{\varepsilon}$. We need three simple lemmas.

Lemma 2.6. If diam $V \leqslant 100 \min(m, k)$, then

$$e^{-\beta h m |V|} Z(V^{(m, \cdot)}) = \Xi(V^{(m, \cdot)}) \tag{2.28}$$

Proof. The lemma follows from the definition of the elementary cylinder and from Lemma 2.2. ∎

Lemma 2.7. For any finite volume V

$$e^{-\beta h m |V|} Z(V^{(m, \cdot)}) \leqslant \Xi(V^{(m, \cdot)}) \tag{2.29}$$

Proof. The lemma follows from the positivity of $\tilde{w}(\varepsilon)$, $\tilde{w}(\gamma)$, from the definition of $Z(\cdot)$, and from Lemma 2.2. ∎

Lemma 2.8. For any finite volume V and $0<m<n$

$$\frac{\Xi(V^{(m,\cdot)})}{\Xi(V^{(n,\cdot)})}\leqslant e^{\beta h(n-m)\,|V|} \tag{2.30}$$

Proof.

$$\begin{aligned}\Xi(V^{(m)})&=\sum_{\phi_V}\exp\{-\beta H(\phi_V\mid\phi_{V^c}^{(m)})\}\\&=\exp\{\beta h(n-m)\,|V|\}\sum_{\phi_V}\exp\{-\beta H(\phi_V+n-m\mid\phi_{V^c}^{(n)})\}\\&\leqslant\exp\{\beta h(n-m)\,|V|\}\;\Xi(V^{(n)})\end{aligned}$$

Partition functions $\Xi(V^{(m,\cdot)})$ and $\Xi(V^{(n,\cdot)})$ can be compared analogously. ▌

Now we apply these lemmas and consider first $\varepsilon=(\tilde{\varepsilon},E,I)$ with $S(\varepsilon)=1$. Due to Lemma 2.6

$$\Xi(\bar{\varepsilon}^{(E,S)})=\exp(-\beta hE\,|\bar{\varepsilon}|)\,Z(\bar{\varepsilon}^{(E,S)}),\qquad\Xi(\bar{\varepsilon}^{(I,S)})=\exp(-\beta hI\,|\bar{\varepsilon}|)\,Z(\bar{\varepsilon}^{(I,S)})$$

Hence the induction hypothesis (ii) can be applied to get the bounds

$$\begin{aligned}&\exp\left\{\frac{1}{2}\,|\bar{\varepsilon}|\exp(-4\beta E)-\beta hL\,|\bar{\varepsilon}|-|\bar{\varepsilon}|\exp\left(-4\beta\min(E,k)-\frac{\beta}{100}\right)\right\}\\&\quad\leqslant\frac{\Xi(\bar{\varepsilon}^{(I,S)})}{\Xi(\bar{\varepsilon}^{(E,S)})}\\&\quad\leqslant\exp\left\{2\,|\bar{\varepsilon}|\exp(-4\beta E)-\beta hL\,|\bar{\varepsilon}|+|\bar{\varepsilon}|\exp\left(-4\beta\min(E,k)-\frac{\beta}{100}\right)\right\}\end{aligned} \tag{2.31}$$

If $L\leqslant 100k$, then

$$2\,|\bar{\varepsilon}|\exp(-4\beta E)+\beta hL\,|\bar{\varepsilon}|+|\bar{\varepsilon}|\exp\left\{-4\beta\min(E,k)-\frac{\beta}{100}\right\}\leqslant\frac{1}{100}$$

because $\operatorname{diam}\bar{\varepsilon}\leqslant 100\min(E,k)$ and $\beta h\leqslant e^{-4\beta(k-1)}$. Hence

$$\begin{aligned}|w_1(\varepsilon)|&\leqslant\exp(-\beta\,|\bar{\varepsilon}|\,L)\,4\,|\bar{\varepsilon}|\,\{L\exp(-4\beta(k-1))+\exp(-4\beta E)\}\\&\leqslant\exp\left(-\frac{\beta}{100}\,|\bar{\varepsilon}|\,L\right)\exp\{-4\beta\min(E,k)\}\\&\qquad\times\left\{8L\,|\bar{\varepsilon}|\exp\left(-99\,\frac{\beta}{100}\,|\tilde{\varepsilon}|\,L+4\beta\right)\right\}\\&\leqslant\exp\left\{-\frac{\beta}{100}\,|\tilde{\varepsilon}|\,L-4\beta\min(E,k)\right\}\end{aligned}$$

because $|\tilde{\varepsilon}| > 4$. In the opposite case, if $L > 100k$, then

$$\tilde{w}(\varepsilon) \leqslant \exp\left\{-\beta\,|\tilde{\varepsilon}|\,L - \beta hL\,|\tilde{\varepsilon}| + 2\,|\tilde{\varepsilon}|\exp(-4\beta E) + |\tilde{\varepsilon}|\exp\left(-4\beta\min(E,k) - \frac{\beta}{100}\right)\right\}$$

$$\leqslant \exp\left\{-\beta\,|\tilde{\varepsilon}|\,L + \frac{1}{100}\right\}$$

$$\leqslant \exp\left\{-\frac{\beta}{100}\,|\tilde{\varepsilon}|\,L - 4\beta k\right\}$$

$$\leqslant \exp\left\{-\frac{\beta}{100}\,|\tilde{\varepsilon}|\,L - 4\beta\min(E,k)\right\}$$

The case $S(\varepsilon) = -1$ is slightly more complicated. If $L \geqslant 100k$, then due to Lemma 2.8

$$\tilde{w}(\varepsilon) = \exp(-\beta\,|\tilde{\varepsilon}|\,L)\,\frac{\Xi(\tilde{\varepsilon}^{(I,S)})}{\Xi(\tilde{\varepsilon}^{(E,S)})}$$

$$\leqslant \exp\{-\beta\,|\tilde{\varepsilon}|\,L + \beta hL\,|\tilde{\varepsilon}|\}$$

$$\leqslant \exp\{-\beta\,|\tilde{\varepsilon}|\,L + L\}$$

$$\leqslant \exp\left\{-\frac{\beta}{100}\,|\tilde{\varepsilon}|\,L - 4\beta k\right\}$$

$$\leqslant \exp\left\{-\frac{\beta}{100}\,|\tilde{\varepsilon}|\,L - 4\beta\min(E,k)\right\}$$

Here we use the bound $|\tilde{\varepsilon}|\,\beta h \leqslant |\tilde{\varepsilon}|\,e^{-4\beta(k-1)} \leqslant 1$, which follows from the condition $\operatorname{diam}\tilde{\varepsilon} \leqslant 100\min(E,k)$.

If $L < 100k$ and $|\tilde{\varepsilon}|\,e^{-4\beta I} > 1/100$, then

$$|\tilde{\varepsilon}| > \frac{1}{100}\,e^{4\beta I}$$

and hence

$$\frac{1}{10}\,e^{2\beta I} \leqslant \operatorname{diam}\tilde{\varepsilon} \leqslant 100E$$

Therefore

$$E > \frac{1}{1000}\,e^{2\beta I} \qquad \text{and} \qquad L = E - I > \frac{9}{10}\,E$$

Again by Lemma 2.8

$$\begin{aligned}\tilde{w}(\varepsilon) &\leqslant \exp\{-\beta\,|\bar{\varepsilon}|\,L+\beta hL\,|\bar{\varepsilon}|\}\\ &\leqslant \exp\{-\beta\,|\bar{\varepsilon}|\,L+L\}\\ &\leqslant \exp\left\{-\frac{\beta}{100}\,|\bar{\varepsilon}|\,L\right\}\exp\left\{-98\,\frac{\beta}{100}\,|\bar{\varepsilon}|\,\frac{9}{10}\,E\right\}\\ &\leqslant \exp\left\{-\frac{\beta}{100}\,|\bar{\varepsilon}|\,L-4\beta E\right\}\\ &\leqslant \exp\left\{-\frac{\beta}{100}\,|\bar{\varepsilon}|\,L-4\beta\min(E,k)\right\}\end{aligned}$$

where the obvious estimation $(98/100)(9/10)\,|\bar{\varepsilon}|\geqslant 4$ for $|\bar{\varepsilon}|\geqslant 6$ was used.

Finally, if $L<100k$ and $|\bar{\varepsilon}|\,e^{-4\beta I}<1/100$, then

$$2\,|\bar{\varepsilon}|\,e^{-4\beta E}+\beta hL\,|\bar{\varepsilon}|+2\,|\bar{\varepsilon}|\,e^{-4\beta I}\leqslant 4\,|\bar{\varepsilon}|\,e^{-4\beta I}+|\bar{\varepsilon}|\;100ke^{-4\beta(k-1)}\leqslant\frac{5}{100}$$

Therefore by Lemma 2.8

$$w_1(\varepsilon)\leqslant \exp(-\beta\,|\bar{\varepsilon}|\,L)\,2\,|\bar{\varepsilon}|\,\beta hL\leqslant \exp(-\beta\,|\bar{\varepsilon}|\,L)\,2\,|\bar{\varepsilon}|\,L\exp\{-4\beta(k-1)\}$$

On the other hand

$$\Xi(\bar{\varepsilon}^{(E,S)})=\exp(-\beta hE\,|\bar{\varepsilon}|)\,Z(\bar{\varepsilon}^{(E,S)})$$

by Lemma 2.6,

$$\Xi(\bar{\varepsilon}^{(I,S)})\geqslant \exp(-\beta hI\,|\bar{\varepsilon}|)\,Z(\bar{\varepsilon}^{(I,S)})$$

by Lemma 2.7, and

$$\begin{aligned}\frac{\Xi(\bar{\varepsilon}^{(I,S)})}{\Xi(\bar{\varepsilon}^{(E,S)})}&\geqslant\frac{\exp(-\beta hI\,|\bar{\varepsilon}|)\,Z(\bar{\varepsilon}^{(I,S)})}{\exp(-\beta hE\,|\bar{\varepsilon}|)\,Z(\bar{\varepsilon}^{(E,S)})}\\ &\geqslant \exp\left\{-2\,|\bar{\varepsilon}|\exp(-4\beta I)+\beta hL\,|\bar{\varepsilon}|\right.\\ &\qquad\left.-\,|\bar{\varepsilon}|\exp\left(-4\beta\min(I,k)-\frac{\beta}{100}\right)\right\}\end{aligned}$$

by the induction hypothesis (ii). Hence,

$$\begin{aligned}w_1(\varepsilon)\geqslant &-\exp(-\beta\,|\bar{\varepsilon}|\,L)\,2\,|\bar{\varepsilon}|\\ &\cdot\left|\beta hL-2\exp(-4\beta I)-\exp\left(-4\beta\min(I,k)-\frac{\beta}{100}\right)\right|\end{aligned}$$

Now, if $I \leqslant k$, then

$$\begin{aligned}
|w_1(\varepsilon)| &\leqslant \exp(-\beta\,|\bar{\varepsilon}|\,L)\,5\,|\bar{\varepsilon}|\,L\exp(-4\beta I)\\
&\leqslant \exp\left(-\frac{\beta}{100}\,|\bar{\varepsilon}|\,L-4\beta E\right)\left\{\exp\left(-\frac{\beta}{100}\,|\bar{\varepsilon}|\,L\right)5L\,|\bar{\varepsilon}|\right\}\\
&\quad\times\exp\left\{-\beta\left(\frac{98}{100}-\frac{4}{|\bar{\varepsilon}|}\right)|\bar{\varepsilon}|\,L\right\}\\
&\leqslant \exp\left\{-\frac{\beta}{100}\,|\bar{\varepsilon}|\,L-4\beta E\right\}\\
&\leqslant \exp\left\{-\frac{\beta}{100}\,|\bar{\varepsilon}|\,L-4\beta\min(E,k)\right\}
\end{aligned}$$

and, if $I > k$, then

$$\begin{aligned}
|w_1(\varepsilon)| &\leqslant \exp(-\beta\,|\bar{\varepsilon}|\,L)\,5\,|\bar{\varepsilon}|\,L\exp\{-4\beta(k-1)\}\\
&\leqslant \exp\left\{-\frac{\beta}{100}\,|\bar{\varepsilon}|\,L-4\beta k\right\}\\
&\leqslant \exp\left\{-\frac{\beta}{100}\,|\bar{\varepsilon}|\,L-4\beta\min(E,k)\right\}
\end{aligned}$$

Thus condition (i) is reproduced.

To finish the proof of Lemma 2.5, let us repeat once more the order in which our induction goes. Supposing (i) to be proven for all ε with $|\bar{\varepsilon}| \leqslant l$ and (ii) to be proven for all $V^{(\cdot)}$ with $|V| \leqslant l-1$ and $V^{(\cdot,\pm)}$ with $|V| \leqslant l$, we first reproduce (ii) for $V^{(\cdot)}$ with $|V| = l$ and for $V^{(\cdot,\pm)}$ with $|V| = l+1$. Then using this fact, we reproduce (i) for ε with $|\bar{\varepsilon}| = l+1$. ∎

Lemma 2.5 contains the estimations of $\tilde{w}(\varepsilon)$, but it also allows us to estimate the statistical weight of contours.

Lemma 2.9. For any contour $\Gamma = \{\gamma^{\text{ext}}, \gamma_i, \gamma^{\text{int},j}\}$

$$\begin{aligned}
w(\Gamma) \leqslant \exp\Big(&-\beta\,|\tilde{\gamma}^{\text{ext}}|\,L(\gamma^{\text{ext}})-\beta\sum_i|\tilde{\gamma}_i|\,L(\gamma_i)-\beta\sum_j|\tilde{\gamma}^{\text{int},j}|\,L(\gamma^{\text{int},j})\\
&-\tfrac{1}{3}\,|\text{Supp}_e(\Gamma)|\exp\{-4\beta\min(I(\gamma^{\text{ext}}),k)\}\\
&-\tfrac{1}{3}\sum_i|\text{Supp}_i(\Gamma)|\exp\{-4\beta\min(I(\gamma_i),k)\}\Big)
\end{aligned}\tag{2.32}$$

Proof. If we substitute the obvious bound

$$Z(\mathrm{Supp}(\Gamma)^{(k,\dots)}) \geqslant Z(\mathrm{Supp}_e(\Gamma)^{(k,\dots)}) \prod_i Z(\mathrm{Supp}_i(\Gamma)^{(k,\dots)})$$

into the definition (2.14) and apply Lemma 2.5 to every term,

$$\frac{e^{-\beta h J(\gamma_i)\,|\mathrm{Supp}_i(\Gamma)|} Z(\mathrm{Supp}_i(\Gamma)^{(J(\gamma_i),\dots)})}{e^{-\beta h k\,|\mathrm{Supp}_i(\Gamma)|} Z(\mathrm{Supp}_i(\Gamma)^{(k,\dots)})}$$

then we immediately obtain Lemma 2.9. ∎

Now we are ready to construct a cluster expansion for $\ln \Xi(V^{(k,\cdot)})$. Here we use a scheme proposed in ref. 8. The cluster representation (2.15) we rewrite now as

$$\Xi(V^{(k,\cdot)}) = e^{-\beta h k\,|V|} \sum_{[\Gamma_l;\varepsilon_m] \in V^{(k,\cdot)}} \prod_l w(\Gamma_l) \prod_m \tilde{w}(\varepsilon_m) \tag{2.33}$$

where the sum is extended over compatible collections of *clusters* (i.e., contours and elementary cylinders) belonging to the volume $V^{(k,\cdot)}$. The collection $[\Gamma_l; \varepsilon_m]$ is called a *compatible collection of clusters* if:

(i) $\mathrm{Supp}(\Gamma_{l'}) \cap \mathrm{Supp}(\Gamma_{l''}) = \varnothing$ for any $\Gamma_{l'}$ and $\Gamma_{l''}$.

(ii) Any $\varepsilon_{m'}$ and $\varepsilon_{m''}$ are weakly compatible.

(iii) For any Γ_l and ε_m the boundary $\partial\,\mathrm{Supp}(\Gamma_l)$ and ε_m are weakly compatible.

(iv) $E(\Gamma_l) = E(\varepsilon_m) = k$ for any l and m.

For our purposes the result of ref. 8 can be formulated as follows.

Lemma 2.10. If there exist functions $a(\varepsilon)$ and $b(\Gamma)$ such that

$$\sum_{\varepsilon':\,\tilde{\varepsilon}' \cap \partial\,\mathrm{Supp}(\Gamma) \neq \varnothing} \tilde{w}(\varepsilon')\, e^{a(\varepsilon')} + \sum_{\Gamma':\,\mathrm{Supp}(\Gamma') \cap \mathrm{Supp}(\Gamma) \neq \varnothing} w(\Gamma')\, e^{b(\Gamma')} \leqslant b(\Gamma) \tag{2.34}$$

and

$$\sum_{\varepsilon':\,\tilde{\varepsilon}' \cap \tilde{\varepsilon} \neq \varnothing} \tilde{w}(\varepsilon')\, e^{a(\varepsilon')} + \sum_{\Gamma':\,\partial\,\mathrm{Supp}(\Gamma') \cap \tilde{\varepsilon} \neq \varnothing} w(\Gamma')\, e^{b(\Gamma')} \leqslant a(\varepsilon) \tag{2.35}$$

then

$$\ln \Xi(V^{(k,\cdot)}) = \sum_{\xi \in V^{(k,\cdot)}} w(\xi) \tag{2.36}$$

where the series on the RHS of (2.36) is absolutely convergent, the statistical weight $w(\xi)$ of polymers ξ is invariant under translations of $\mathbf{Z}^2$, and polymers $\xi = (\Gamma_i; \varepsilon_j)$ are the collections of clusters with the following property: any two elements of ξ can be joined by the sequence of elements such that every two neighboring clusters in this sequence are not compatible.

Proof. See ref. 8. ▮

Corollary. If the cluster expansion (2.36) is valid for $\ln \Xi(V^{(k,\cdot)})$, then there exists the limit Gibbs state generated by the boundary condition $\phi^{(k)}$.

Proof. The limit probabilities of any compatible collection of clusters can be directly written in terms of $w(\Gamma)$, $\tilde{w}(\varepsilon)$, and $w(\xi)$.[10] ▮

Lemma 2.11. Let

$$a(\varepsilon) = |\bar{\varepsilon}|\, e^{-\beta/200} \tag{2.37}$$

and

$$b(\Gamma) = \frac{1}{100}\, |\mathrm{Supp}(\Gamma)|\, e^{-4\beta k} + |\partial\, \mathrm{Supp}(\Gamma)|\, e^{-\beta/200} \tag{2.38}$$

Then $a(\varepsilon)$ and $b(\Gamma)$ satisfy (2.34)–(2.35).

Proof. The following two lemmas are important for our proof.

Lemma 2.12. For any u, $t > 0$

$$\sum_{\gamma:\, \tilde{\gamma} \ni 0,\, |\tilde{\gamma}| > t} \exp\{-u\beta\, |\tilde{\gamma}|\, L(\gamma)\} \leqslant \exp(-\tfrac{1}{2} ut\beta) \tag{2.39}$$

$$\sum_{\gamma:\, \tilde{\gamma} \ni 0,\, |\tilde{\gamma}| > t} \exp\{-u\beta\, |\tilde{\gamma}|\, L(\gamma)\} \leqslant \exp(-\tfrac{1}{2} ut\beta) \tag{2.40}$$

Proof. Due to the connectedness of $\tilde{\gamma}$ the number of $\tilde{\gamma}$ with $|\tilde{\gamma}| = n$ and $\tilde{\gamma} \ni 0$ is less than e^{cn}. So

$$\sum_{\gamma:\, \tilde{\gamma} \ni 0,\, |\tilde{\gamma}| > t} \exp\{-u\beta\, |\tilde{\gamma}|\, L(\gamma)\} \leqslant \sum_{n=t}^{\infty} \exp(cn) \sum_{l=1}^{\infty} \exp(-u\beta n l) \leqslant \exp(-\tfrac{1}{2} ut\beta)$$

If $\bar{\gamma} \ni 0$ instead of $\tilde{\gamma} \ni 0$ and $|\tilde{\gamma}| = n$, then there exists a point $(0, q) \in \mathbf{Z}^2$, $|q| \leqslant n$, which belongs to $\tilde{\gamma}$. Hence, the number of such $\tilde{\gamma}$ is less than $2ne^{cn}$ and we come to the same answer. ▮

Bounds (2.39) and (2.40) are quite standard and, in fact, we already used them several times during the proof of Lemma 2.5.

Lemma 2.13. Consider a treelike graph T with the vertices v labeled by an integer parameter n_v and with the edges g labeled by two integer parameters l_g and n_g. Denote by $d(v)$ the number of edges incident to the vertex v and suppose that:

(i) $d(v) < n_v$.

(ii) If some edge g leads to the vertex v', then $100 n_g \leqslant n_{v'}$.

Assign to every tree T the statistical weight

$$w(T) = \prod_{v \in T} C_{n_v}^{d(v)} \exp(-\beta n_v) \prod_{g \in T} \exp\{-l_g \exp(-\beta n_g)\} \tag{2.41}$$

where $C_{n_v}^{d(v)}$ is the binomial coefficient. Then

$$\sum_{T:\, n_{r(T)} \geqslant t} w(T) \leqslant e^{-t\beta/2} \tag{2.42}$$

where $r(T)$ denotes the root of T.

Proof. We proceed by the induction on number of vertices in T. Suppose that (2.42) is proven for all T with the number of vertices $|T|$ less than q. Any tree T with $|T| = q$ can be decomposed into root r, edges $g_1,..., g_f$ which lead from the root to the vertices $v_1,..., v_f$, and subtrees $T_1,..., T_f$ such that $|T_i| < q$ and T_i has v_i as the root. Then

$$\begin{aligned}
&\sum_{T:\, |T| = q,\, n_r \geqslant t} w(T) \\
&\quad \leqslant \sum_{n_r = t}^{\infty} \exp(-\beta n_r) \sum_{f=1}^{n_r} C_{n_r}^{f} \\
&\qquad \times \prod_{i=1}^{f} \Bigg(\sum_{n_{v_i} = 100}^{\infty} \sum_{l_i = 1}^{\infty} \sum_{n_i = 1}^{[n_{v_i}/100]} \exp\{-l_i \exp(-\beta n_i)\} \\
&\qquad \times \sum_{T_i:\, |T_i| \leqslant q-1,\, r(T_i) = v_i} w(T) \Bigg) \\
&\quad \leqslant \sum_{n_r = t}^{\infty} \exp(-\beta n_r) \sum_{f=1}^{n_r} C_{n_r}^{f} \\
&\qquad \times \prod_{i=1}^{f} \Bigg(\sum_{n_{v_i} = 100}^{\infty} \sum_{l_i = 1}^{\infty} \sum_{n_i = 1}^{[n_{v_i}/100]} \exp\{-l_i \exp(-\beta n_i)\} \exp(-\tfrac{1}{2} n_{v_i} \beta) \Bigg)
\end{aligned}$$

$$= \sum_{n_r=t}^{\infty} \exp(-\beta n_r) \sum_{f=1}^{n_r} C_{n_r}^f$$

$$\times \prod_{i=1}^{f} \left(\sum_{n_i=1}^{\infty} \sum_{l_i=1}^{\infty} \exp\{-l_i \exp(-\beta n_i)\} \sum_{n_{v_i}=100n_i}^{\infty} \exp(-\tfrac{1}{2} n_{v_i} \beta) \right)$$

$$\leqslant \sum_{n_r=t}^{\infty} \exp(-\beta n_r) \sum_{f=1}^{n_r} C_{n_r}^f$$

$$\times \prod_{i=1}^{f} \left(\sum_{n_i=1}^{\infty} \sum_{l_i=1}^{\infty} \exp\{-l_i \exp(-\beta n_i)\} \exp(-40 n_i \beta) \right)$$

$$\leqslant \sum_{n_r=t}^{\infty} \exp(-\beta n_r) \sum_{f=1}^{n_r} C_{n_r}^f \prod_{i=1}^{f} \left\{ \sum_{n_i=1}^{\infty} \exp(\beta n_i) \exp(-35 n_i \beta) \right\}$$

$$\leqslant \sum_{n_r=t}^{\infty} \exp(-\beta n_r) \sum_{f=1}^{n_r} C_{n_r}^f \prod_{i=1}^{f} \exp(-30\beta)$$

$$\leqslant \sum_{n_r=t}^{\infty} \exp(-\beta n_r) \sum_{f=1}^{n_r} C_{n_r}^f \exp(-30 f \beta)$$

$$\leqslant \sum_{n_r=t}^{\infty} \exp(-\beta n_r) \{1 + \exp(-30\beta)\}^{n_r} \leqslant \exp(-\tfrac{1}{2} t \beta)$$

For T with $|T| = 1$ lemma is obvious. ∎

Returning to the proof of Lemma 2.11, we apply Lemma 2.12 and obtain the bound

$$\sum_{\varepsilon' : \tilde{\varepsilon}' \cap \tilde{\varepsilon} \neq \varnothing} \tilde{w}(\varepsilon') \exp\{a(\varepsilon')\}$$

$$\leqslant \sum_{\varepsilon' : \tilde{\varepsilon}' \cap \tilde{\varepsilon} \neq \varnothing} \exp\left\{ -\frac{\beta}{100} |\tilde{\varepsilon}'| \, L(\varepsilon') \right\} \exp\left\{ |\tilde{\varepsilon}'| \exp\left(-\frac{\beta}{200} \right) \right\}$$

$$\leqslant |\tilde{\varepsilon}| \sum_{\varepsilon' : \tilde{\varepsilon}' \ni 0} \exp\left\{ -\frac{\beta}{150} |\tilde{\varepsilon}'| \, L(\varepsilon') \right\} \leqslant \frac{1}{2} |\tilde{\varepsilon}| \exp\left(-\frac{\beta}{200} \right)$$

Similarly,

$$\sum_{\varepsilon' : \tilde{\varepsilon}' \cap \partial \operatorname{Supp}(\Gamma) \neq \varnothing} \tilde{w}(\varepsilon') \, e^{a(\varepsilon')} \leqslant |\partial \operatorname{Supp}(\Gamma)| \, e^{-\beta/200}$$

The estimation of other two terms entering in (2.34) and (2.35) is less trivial. Here the key role is played by the following lemma.

Lemma 2.14. Let

$$\tilde{w}(\Gamma)=\exp\Big(-\tfrac{1}{2}\beta\,|\tilde{\gamma}^{\mathrm{ext}}|\,L(\gamma^{\mathrm{ext}})-\tfrac{1}{2}\beta\sum_i|\tilde{\gamma}_i|\,L(\gamma_i)$$

$$-\tfrac{1}{2}\beta\sum_j|\tilde{\gamma}^{\mathrm{int},j}|\,L(\gamma^{\mathrm{int},j})$$

$$-\tfrac{1}{4}\,|\mathrm{Supp}_e(\Gamma)|\,\exp\{-4\beta\min(I(\gamma^{\mathrm{ext}}),k)\}$$

$$-\tfrac{1}{4}\sum_i|\mathrm{Supp}_i(\Gamma)|\,\exp\{-4\beta\min(I(\gamma_i),k)\}\Big)\tag{2.43}$$

Then

$$\sum_{\Gamma:\,\tilde{\gamma}^{\mathrm{ext}}(\Gamma)\ni 0}\tilde{w}(\Gamma)\leqslant e^{-100\beta k}\tag{2.44}$$

Proof. Consider more carefully the geometry of a given contour $\Gamma=\{\gamma^{\mathrm{ext}},\gamma_i,\gamma^{\mathrm{int},j}\}$. The statistical weight $\exp\{-\frac{1}{2}\beta\,|\tilde{\gamma}|\,L(\gamma)\}$ is concentrated on every cylinder of the contour Γ, but unfortunately the set of bases of these cylinders is disconnected. We apply Lemma 2.13 to show that nevertheless the statistical weights $\tilde{w}(\Gamma)$ are summable. For this purpose we complete the set of bases of cylinders to the connected structure in the following way. The interiors of cylinders from Γ are partially ordered by inclusion. Choose the minimal ones in the sense of this partial order and fix an arbitrary point from $\tilde{\mathbf{Z}}^2$ in the interior of every minimal cylinder. (Note that $\gamma^{\mathrm{int},j}$ are the minimal elements, but in general not the only ones.) Draw inside $\mathrm{Supp}(\Gamma)$ the lines parallel to the first coordinate axis of $\tilde{\mathbf{Z}}^2$ passing trough all previously fixed points. The union of the lines and the bases of the cylinders forms a connected subset of $\tilde{\mathbf{Z}}^2$. The bases of cylinders of Γ cut lines into line segments. Some segments can be deleted from our connected structure without destroying its connectedness and we delete all of them. Consider two cylinders not separated by another cylinder. The only line segment which passes from the base of the first cylinder to the base of the second cylinder belongs to some $\mathrm{Supp}_{(\cdot)}(\Gamma)$ and for the different pairs of cylinders these segments are mutually disjoint.

Now we spread the statistical weight $\tilde{w}(\Gamma)$ between the elements of the connected structure just constructed. We assign the statistical weight $\exp\{-\frac{1}{2}\beta\,|\tilde{\gamma}|\,L(\gamma)\}$ to every cylinder of Γ and we assign the statistical weight $\exp\{-\frac{1}{4}\,|p|\exp(-4\beta\min(I(\gamma_i),k))\}$ to every line segment $p\in\mathrm{Supp}_i(\Gamma)$ of length $|p|$. Obviously the product of these statistical weights taken over all cylinders and all line segments of Γ remains less than or equal to $\tilde{w}(\Gamma)$ [due to construction the sum of the lengths of line segments is less than $|\mathrm{Supp}(\Gamma)|$].

Let $\gamma' \prec \gamma''$ be the notation for any pair of cylinders γ' and γ'' not separated by the third cylinder and with $\bar{\gamma}' \subset \bar{\gamma}''$. The observation which allows us to apply Lemma 2.13 is the following. If some segment p joins $\tilde{\gamma}_{i'}$ and $\tilde{\gamma}_{i''}$ for $\gamma_{i''} \prec \gamma_{i'}$, then the statistical weight of p is

$$\exp\{-\tfrac{1}{4}|p| \exp(-4\beta \min(I(\gamma_{i'}), k))\}$$

while

$$\operatorname{diam} \tilde{\gamma}_{i''} \geqslant 100 \min(I(\gamma_{i'}), k)$$

If some segment p joins $\tilde{\gamma}_{i'}$ and $\tilde{\gamma}_{i''}$ with $\gamma_{i'}, \gamma_{i''} \prec \gamma_{i'''}$ and $\bar{\gamma}_{i'} \cap \bar{\gamma}_{i''} = \varnothing$, then the statistical weight of p is

$$\exp\{-\tfrac{1}{4}|p| \exp(-4\beta \min(I(\gamma_{i'''}), k))\}$$

while

$$\operatorname{diam} \tilde{\gamma}_{i''} \geqslant 100 \min(I(\gamma_{i'''}), k)$$

Up to nonessential details we fit now into the conditions of Lemma 2.13. The cylinders $\gamma \in \Gamma$ correspond to vertices v of the tree with $n_v = |\tilde{\gamma}|$ and segments $p \in \mathrm{Supp}_i(\Gamma)$ correspond to edges g with $l_g = |p|$, $n_g = \min(I(\gamma_i), k)$. The binomial coefficient $C_{n_v}^{d(v)}$ counts the number of possibilities to choose the starting points of the segments beginning from a given cylinder. The cylinder γ^{ext} corresponds to the root of the tree and by our construction $\operatorname{diam} \tilde{\gamma}^{\mathrm{ext}} > 100k$.

This identification proves Lemma 2.14. ∎

Applying Lemma 2.14, we obtain

$$\begin{aligned}
\sum_{\Gamma' :\, \mathrm{Supp}(\Gamma') \cap \mathrm{Supp}(\Gamma) \neq \varnothing} w(\Gamma')\, e^{b(\Gamma')} &\leqslant \sum_{\Gamma' :\, \mathrm{Supp}(\Gamma') \cap \mathrm{Supp}(\Gamma) \neq \varnothing} \tilde{w}(\Gamma') \\
&\leqslant \sum_{x \in \mathrm{Supp}(\Gamma)} \sum_{\Gamma' :\, \tilde{\gamma}^{\mathrm{ext}}(\Gamma') \ni x} \tilde{w}(\Gamma') \\
&\leqslant \sum_{x \in \mathrm{Supp}(\Gamma)} e^{-100\beta k} \leqslant \frac{1}{100} |\mathrm{Supp}(\Gamma)|\, e^{-4\beta k}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{\Gamma' :\, \partial\, \mathrm{Supp}(\Gamma') \cap \tilde{\varepsilon} \neq \varnothing} w(\Gamma')\, e^{b(\Gamma')} &\leqslant \sum_{x \in \tilde{\varepsilon}} \sum_{\Gamma' :\, \tilde{\gamma}^{\mathrm{ext}}(\Gamma') \ni x} \tilde{w}(\Gamma') \\
&\leqslant |\tilde{\varepsilon}|\, e^{-100\beta k} \leqslant \frac{1}{2} |\tilde{\varepsilon}|\, e^{-\beta/200}
\end{aligned}$$

which proves Lemma 2.11. ∎

To finish the proof of Theorem 2.2, it is necessary to establish the uniqueness of the limit Gibbs measure just constructed. For this purpose consider a large square volume V with an arbitrary boundary condition ϕ_{V^c}. Let us denote such a volume as $V^{(\phi)}$. For any compatible collection of clusters $[\Gamma_l; \varepsilon_m] \in V^{(\phi)}$ the statistical weight of the clusters not touching ∂V is the same as for the standard boundary condition $\phi^{(k)}$ but the statistical weight of the clusters touching ∂V is changed depending on ϕ_{V^c}. If some cluster touches ∂V by q bonds such that along these bonds $\phi = \phi^{(n)}$, $n \neq k$, then the additional factor $\exp(2\beta q\,|n-k|)$ enters the statistical weight of contour Γ_l (or elementary cylinder ε_m) with $\text{sign}(n-k) = S(\gamma^{\text{ext}}(\Gamma_l))$ [or $\text{sign}(n-k) = S(\varepsilon_m)$]. So we join all clusters which touch ∂V into one object. This union of clusters we denote by Ω and we call it a boundary cluster. Let $\text{Supp}(\Omega)$ be the union of the supports of contours of Ω and let $\tilde{\Omega}$ be the union of the bases of elementary cylinders of Ω. The boundary cluster Ω generates the boundary condition $\phi^{(k)}$ on the volume $V \backslash \text{Supp}(\Omega)$. Hence the results of ref. 15 imply that for the uniqueness in the class of $\mathbf{Z}^2$-periodic limit Gibbs states it is sufficient to verify the following lemma.

Lemma 2.15. For an arbitrary boundary condition ϕ_{V^c} and for some constant $0 < \alpha < 1$ the probability of the event $\{|\text{Supp}(\Omega)| > V^\alpha\}$ tends to zero as $V \nearrow \mathbf{Z}^2$.

Proof. The probability of any boundary cluster Ω is less than its statistical weight. The statistical weight of Ω is the product of the statistical weights of elementary cylinders $\varepsilon_m(\Omega)$ and the statistical weights of contours $\Gamma_l(\Omega)$. The modified statistical weight of any $\varepsilon_m(\Omega)$ remains less than or equal to 1. For any contour $\Gamma_l(\Omega)$ its modified statistical weight is less than

$$\exp\{2\beta\,|\tilde{\gamma}^{\text{ext}}(\Gamma_l) \cap \partial V|\; L(\gamma^{\text{ext}}(\Gamma_l))\}\; w(\Gamma_l)$$

Therefore it follows from the proof of Lemmas 2.13 and 2.14 that the probability of Ω with fixed $|\text{Supp}(\Omega)|$ is less than

$$\exp\{2\beta \max_{x \in \mathbf{Z}^2} (\phi_x)\,|\partial V| - \tfrac{1}{8}\,|\text{Supp}(\Omega)| \exp(-4\beta k)\}$$

which proves the lemma. ∎

Let us now return to the proof of Theorem 2.1. The main difference between the two theorems is the following. For

$$\frac{1}{\beta}\exp\left(-4\beta k - \frac{\beta}{100}\right) \leqslant h \leqslant \frac{1}{\beta}\exp\left(-4\beta k + \frac{\beta}{100}\right) \qquad (2.45)$$

Lemma 2.5 implies that the partition functions

$$e^{-\beta hk|V|}Z(V^{(k,\cdot)}), \qquad e^{-\beta h(k+1)|V|}Z(V^{(k+1,\cdot)})$$

are approximately equal and exceed $e^{-\beta hm|V|}Z(V^{(m,\cdot)})$ for $m \neq k$, $k+1$. So we have two dominant ground states, $\phi^{(k)}$ and $\phi^{(k+1)}$, and one can apply the Pirogov–Sinai theory in the spirit of ref. 15. Here we present the sketch of the proof only, because the whole proof is quite standard and tedious.

At first we construct $h_k^*(\beta)$. For this purpose we complete the set of elementary cylinders by all cylinders $\eta = (\tilde{\eta}, k+1, k)$ and $\eta = (\tilde{\eta}, k, k+1)$. Obviously the representation (2.33) remains valid for $\Xi(V^{(k,\cdot)})$ and $\Xi(V^{(k+1,\cdot)})$, but we do not have a good bound on $\tilde{w}(\eta)$. Hence the cluster expansion cannot be written for $\ln \Xi(V^{(k,\cdot)})$ and $\ln \Xi(V^{(k+1,\cdot)})$ directly. Instead of this we introduce the truncated statistical weight

$$\bar{w}(\eta) = \min(\tilde{w}(\eta), \exp\{-\tfrac{1}{3}\beta\, |\tilde{\eta}|\, L(\eta)\}) \tag{2.46}$$

and the corresponding truncated partition functions $\bar{\Xi}(V^{(k,\cdot)})$ and $\bar{\Xi}(V^{(k+1,\cdot)})$.

The arguments below show that for fixed β the equation

$$\lim_{V \to \mathbf{Z}^2} \frac{1}{|V|} \ln \bar{\Xi}(V^{(k,\cdot)}) = \lim_{V \to \mathbf{Z}^2} \frac{1}{|V|} \ln \bar{\Xi}(V^{(k+1,\cdot)}) \tag{2.47}$$

has a unique solution in segment (2.45). It follows from Theorem 2.2 that for $h = (1/\beta)\exp(-4\beta k + \beta/100)$

$$\bar{\Xi}(V^{(k,\cdot)}) = \Xi(V^{(k,\cdot)}), \qquad \bar{\Xi}(V^{(k+1,\cdot)}) < \Xi(V^{(k+1,\cdot)})$$

and hence

$$a = \lim_{V \to \mathbf{Z}^2} \frac{1}{|V|} \{\ln \bar{\Xi}(V^{(k,\cdot)}) - \ln \bar{\Xi}(V^{(k+1,\cdot)})\} > 0 \tag{2.48}$$

Analogously at the point $h = (1/\beta)\exp(-4\beta k - \beta/100)$

$$\bar{\Xi}(V^{(k+1,\cdot)}) = \Xi(V^{(k+1,\cdot)}), \qquad \bar{\Xi}(V^{(k,\cdot)}) < \Xi(V^{(k,\cdot)})$$

and $a < 0$. According to the cluster expansion of type (2.36) one can write down the representation

$$a = \beta h + \sum_{\xi^k:\, \bar{\xi}^k \ni 0} \frac{w(\xi^k)}{|\bar{\xi}^k|} - \sum_{\xi^{k+1}:\, \bar{\xi}^{k+1} \ni 0} \frac{w(\xi^{k+1})}{|\bar{\xi}^{k+1}|} \tag{2.49}$$

where ξ^k, ξ^{k+1} are polymers for the corresponding truncated models and $\bar{\xi}^k$, $\bar{\xi}^{k+1}$ are their interiors. It can be checked that the derivative $a'_{\beta h}$ is positive for h from segment (2.45) because the derivative of the first term in (2.49) is equal to 1, while the absolute values of derivatives of the last two sums are much less than 1.

By induction on the volume it can be easily shown that

$$\bar{\Xi}(V^{(k+1,\cdot)}) = \Xi(V^{(k+1,\cdot)}), \qquad \bar{\Xi}(V^{(k,\cdot)}) = \Xi(V^{(k,\cdot)})$$

for h being the solution of (2.47). Hence this solution of Eq. (2.47) defines the coexistence curve $h_k^*(\beta)$ of phases $\phi^{(k)}$ and $\phi^{(k+1)}$ on the phase diagram. The proof of the uniqueness of limit Gibbs states generated by the boundary conditions $\phi^{(k)}$ and $\phi^{(k+1)}$ in the class of $\mathbf{Z}^2$-periodic Gibbs states is analogous to the proof of Lemma 2.15.

The case $a > 0$ is slightly more complicated. Again by induction on the volume one can verify that

$$\bar{w}(\eta) = \tilde{w}(\eta), \qquad \bar{\Xi}(\bar{\eta}^{(\cdot,\cdot)}) = \Xi(\bar{\eta}^{(\cdot,\cdot)})$$

for η with

$$\operatorname{diam} \bar{\eta} \leqslant \frac{\beta}{6a} \tag{2.50}$$

Using this fact, it is possible to prove that

$$\bar{w}(\eta) = \tilde{w}(\eta)$$

for any $\eta = (\bar{\eta}, k, k+1)$. Indeed, let η_* be the minimal cylinder with $\bar{w}(\eta_*) \neq \tilde{w}(\eta_*)$. Denote by $\{\eta_i\}^{\text{ext}}$ a compatible collection of external cylinders $\eta_i = (\bar{\eta}_i, k+1, k) \in \bar{\eta}_*^{(k+1,+)}$ with $\operatorname{diam} \bar{\eta}_i > \beta/6a$. The partition function $\Xi(\bar{\eta}_*^{(k+1,+)})$ can be represented as the sum

$$\Xi(\bar{\eta}_*^{(k+1,+)}) = \sum_{\{\eta_i\}^{\text{ext}}} \tilde{\bar{\Xi}}(U^{(k+1,+)}) \prod_i \exp\{-\beta\, |\bar{\eta}_i|\, L(\eta_i)\}\, \Xi(\bar{\eta}_i^{(k,-)}) \tag{2.51}$$

where $U = \bar{\eta}_* \backslash (\bigcup_i \bar{\eta}_i)$ and $\tilde{\bar{\Xi}}(\cdot)$ denotes the partition function over compatible collections of cylinders different from $\eta = (\bar{\eta}, k+1, k)$ with $\operatorname{diam} \bar{\eta} > \beta/6a$. By construction

$$\bar{\Xi}(\bar{\eta}_i^{(k,-)}) = \Xi(\bar{\eta}_i^{(k,-)}), \qquad \bar{\Xi}(\bar{\eta}_*^{(k,+)}) = \Xi(\bar{\eta}_*^{(k,+)})$$

$$\tilde{\bar{\Xi}}(U^{(k+1,+)}) \leqslant \bar{\Xi}(U^{(k+1,+)})$$

Applying Lemmas 2.13 and 2.14, one can easily carry out the summation over $\{\eta_i\}^{\text{ext}}$ and prove that

$$\frac{\Xi(\tilde{\eta}_*^{(k+1,+)})}{\Xi(\tilde{\eta}_*^{(k,+)})} < \sum_{\{\eta_i\}^{\text{ext}}} \frac{\bar{\Xi}(U^{(k+1,+)})}{\bar{\Xi}(U^{(k,+)})} \prod_i \exp\{-\beta\,|\tilde{\eta}_i|\,L(\eta_i)\}$$

$$< \sum_{\{\eta_i\}^{\text{ext}}} \exp\{-a\,|U| + e^{-\beta}\,|\tilde{\eta}_*|\} \prod_i \exp\{-(\beta - e^{-\beta})\,|\tilde{\eta}_i|\,L(\eta_i)\}$$

$$< \exp\{e^{-\beta}\,|\tilde{\eta}_*|\}$$

which contradicts the assumption $\bar{w}(\eta_*) \neq \tilde{w}(\eta_*)$. If $\bar{w}(\eta) = \tilde{w}(\eta)$ for any $\eta = (\tilde{\eta}, k, k+1)$, then

$$\bar{\Xi}(V^{(k,\cdot)}) = \Xi(V^{(k,\cdot)})$$

for any finite volume V. This proves the existence of the limit Gibbs state generated by the boundary condition $\phi^{(k)}$. The proof of the uniqueness of this state in the class of $\mathbf{Z}^2$-periodic Gibbs states is similar to the proof of Lemma 2.15.

The case $a < 0$ can be considered in an analogous way. ∎

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